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On the Range of Certain Pendulum-Type Equations

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Let us consider the BVP

$$\begin{aligned} &mx''(t) + g_1(x'(t)) = f(t), && t \in [0,T] \\ &x(0) = x(T), && x'(0) = x'(T), \end{aligned}$$

where g_1 is a continuous function. The range $\mathscr{R}_1$ of the operator related to this problem is very well known. In this paper we treat the perturbed problem

$$\begin{aligned} &mx''(t) + g_1(x'(t)) + g_0(x(t)) = f(t), && t \in [0,T] \\ &x(0) = x(T), && x'(0) = x'(T), \end{aligned}$$

where g_0 is of pendulum type, showing that, in general, the range of the perturbed operator is not contained in $\mathscr{R}_1$. This points out an important qualitative difference with respect to the case where g_0 is of the Landesmann–Lazer type. On the other hand we prove that if f is small then the mentioned inclusion is true in general. © 2000 Academic Press

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1. INTRODUCTION

In this paper we deal with the periodic boundary value problem

$$\begin{aligned} & mx''(t) + g_1(x'(t)) + g_0(x(t)) = f(t), && t \in [0,T] \\ & x(0) = x(T), && x'(0) = x'(T) \end{aligned} \tag{1}$$

where $m \neq 0$, $g_1, g_0 : \mathbb{R} \to \mathbb{R}$, g_1 is a continuous function and

$$g_0 \text{ is Lipschitz continuous}, \qquad 2\pi\text{-periodic}, \qquad \int_0^{2\pi} g_0(\tau)\, d\tau = 0, \quad [\mathrm{P}]$$

$f \in C_T \equiv C_T^0$, where

$$C_T^k = \{u \in C^k[0,T] : u^{(i)}(0) = u^{(i)}(T), i = 0, \dots, k\}, \qquad k \in \mathbb{N}.$$

Our main purpose is to establish the (possible) relation between the ranges $\mathscr{R}_1$ and $\mathscr{R}$ of the previous problem, for the respective cases $g_0 \equiv 0$ and $g_0 \not\equiv 0$.

We emphasize the qualitative difference between the pendulum-type case, in which g_0 satisfies [P], and the Landesmann–Lazer type case, where

$$\begin{aligned} & g_0 \text{ is bounded, continuous and} \\ & g_0(-\infty) = \lim_{\xi \to -\infty} g_0(\xi) < \lim_{\xi \to +\infty} g_0(\xi) = g_0(+\infty), \end{aligned} \quad [\mathrm{D}]$$

studied previously by Dancer (see [6]).

If we split $f(t) = \tilde{f}(t) + \bar{f}$, where $\bar{f} = \frac{1}{T}\int_0^T f(\tau)\, d\tau$ and

$$\tilde{f} \in \tilde{C}_T = \left\{u \in C_T : \int_0^T u(\tau)\, d\tau = 0\right\},$$

it is known (see [4, Theorem 3.4; 6, Theorem 1]) that if $g_0 \equiv 0$ then for all $\tilde{f}$ there exists an unique $s(\tilde{f}) \in \mathbb{R}$ such that the range of the operator $H_1 : C_T^2 \to C_T$, $H_1(x) = mx'' + g_1(x')$ can be written as

$$\mathscr{R}_1 = \left\{\tilde{f} + s(\tilde{f}) : \int_0^T \tilde{f}(s)\, ds = 0\right\}.$$

Under the assumption [D] it is known (see [6, Theorem 2]) that the range of the operator $H : C_T^2 \to C_T$, $H(x) = mx'' + g_1(x') + g_0(x)$ is

$$\mathscr{R} = \left\{\tilde{f} + \bar{f} : g_0(-\infty) + s(\tilde{f}) < \bar{f} < s(\tilde{f}) + g_0(+\infty)\right\}.$$

Trivially if $g_0(-\infty) < 0 < g_0(+\infty)$ then $\mathscr{R}_1 \subset \mathscr{R}$ (let us note that $g_0(-\infty)$ and $g_0(+\infty) \in \mathbb{R}$).

Consequently one can ask whether the inclusion $\mathscr{R}_1 \subset \mathscr{R}$ holds true also if we consider g_0 satisfying [P]. In the first part of this work we show that the inclusion $\mathscr{R}_1 \subset \mathscr{R}$ is not true in general (see Theorem 2.2). In particular we generalize the result of [10, Theorem B] to equations with nonlinear damping and general Lipschitz continuous periodic nonlinearity representing restoring force of the system. As an extra result we get some new qualitative information about the range of H_1 if g_1 satisfies [P] (see Corollary 2.1).

In the second part we prove that if $\|\tilde{f}\|_{C_T}$ is small enough then $f \in \mathscr{R}_1$ implies $f \in \mathscr{R}$ (see Theorem 4).

Note that in this paper the differential operators work in the spaces of continuous functions, but in [6] Sobolev spaces are considered. It is not hard to show that the result of [6] remains true if we consider the operators H_1 and H working from C_T^2 to C_T. Note that, in [4] g_1 is considered to be bounded, but using an a priori estimate (see [6, 9]), it is possible to remove this limitation, so when we use results from [4] concerning the solvability of (1) with $g_0 = 0$, we will skip the assumption on the boundedness of g_1. In Lemma 2.2 we improve the constant of previous estimations.

To prove the results we use mainly ideas from [4, 6, 10] the Arzelà–Ascoli and the Local Inversion Theorems.

2. MAIN RESULTS

2.1. *Counterexample to* $\mathscr{R}_1 \subset \mathscr{R}$

In this part we state the non-existence of T-periodic solutions for certain periodic BVP and we will use this result to prove the relation between the ranges $\mathscr{R}_1$ and $\mathscr{R}$.

LEMMA 2.1. *Let* $g_0 \not\equiv 0$ *verify* $[P]$ *and* $c \in (0, T \max_{\xi \in [0, 2\pi]} |g_0(\xi)| / 2\pi]$. *Then there exist* $\delta > 0$ *and* $\tilde{f}_0 \in \tilde{C}_T$ (*even there exists* $\tilde{f}_0 \in \tilde{C}_T^\infty$), *such that the equation*

$$cx'(t) + g_0(x(t)) = \tilde{f}_0(t) + \bar{f} \tag{2}$$

does not possess T-periodic solution for all $\bar{f} \in (-\delta, \delta)$.

Moreover there exists a convex, unbounded set $\mathscr{A} \subset \tilde{C}_T$, *with nonempty interior, such that if* $\tilde{f}_0 \in \mathscr{A}$ *and* $\bar{f} = 0$ *then* (2) *does not have a T-periodic solution.*

Proof. At first let us consider $c \in (0, T|\min_{\xi \in [0,2\pi]} g_0(\xi)|/2\pi]$ and define

$$F_1(t) = \frac{\beta_1 t}{c} + \alpha_1, \qquad t \in (0,T), \qquad F_1(t+T) = F_1(t), \text{ a.e. in } \mathbb{R}, \quad (3)$$

where the coefficients α_1, β_1 verify

$$\beta_1 = \min_{\xi \in [0,2\pi]} g_0(\xi) = g_0(\alpha_1).$$

Then $y_1(t) = -\beta_1 t/c$, $t \in [0,T]$, is a solution of

$$cy'(t) + g_0\big(y(t) + F_1(t)\big) = 0 \quad (4)$$

satisfying

$$y(T) \geq y(0) + 2\pi. \quad (5)$$

Now let us see that $cy'(t) + g_0(y(t) + F_1(t)) = 0$ does not admit T-periodic solution. As the function $g_0(y + G_1(t))$ is Lipschitz continuous in y, the Cauchy problem

$$\begin{aligned} &cy'(t) + g_0\big(y(t) + F_1(t)\big) = 0, \qquad t \in \mathbb{R} \\ &y(0) = y_0. \end{aligned} \quad (6)$$

has an unique solution $y(t)$ (see [5, 7]).

If we consider a system of initial conditions $y_n(0) = y_0 + 2n\pi$, where $|y_0| \leq \pi$, we get a system of parallel solutions $y_n(t) = y(t) + 2n\pi$, $\forall n \in \mathbb{N}$. In particular, if $z(t)$ is the solution of the Cauchy problem of (4) with $z(0) = z_0$, there exists a suitable n such that $y_0 + 2(n-1)\pi \leq z_0 \leq y_0 + 2n\pi$. Then we have lower and upper bonds,

$$y(t) + 2(n-1)\pi \leq z(t) \leq y(t) + 2n\pi, \qquad \forall t \in \mathbb{R}. \quad (7)$$

Due to the T-periodicity in t of F_1, $y(t+T)$ is a solution of $cy' + g_0(y + F_1) = 0$, and owing to the 2π-periodicity of g_0, $y(t) + 2\pi$ is also a solution. With respect to (5) and (7) we get $y(t + nT) \geq y(t) + 2n\pi$ for all $t \in \mathbb{R}$, which implies $\lim_{t \to +\infty} y(t) = +\infty$.

Now from (7) it follows that all the solutions of (6) are unbounded, which means that Eq. (4) does not admit T-periodic solutions.

Analogously, let $c \in (0, T|\max_{\xi \in [0,2\pi]} g_0(\xi)|/2\pi]$ and take

$$F_2(t) = \frac{\beta_2 t}{c} + \alpha_2, \qquad t \in (0,T), \qquad F_2(t+T) = F_2(t) \text{ a.e. in } \mathbb{R}, \tag{8}$$

where the coefficients α_2, β_2 are such that

$$\beta_2 = \max_{\xi \in [0,2\pi]} g_0(\xi) = g_0(\alpha_2).$$

Then the solution

$$y_2(t) = -\frac{\beta_2 t}{c}, \qquad t \in [0,T],$$

of the equation

$$cy'(t) + g_0\big(y(t) + F_2(t)\big) = 0 \tag{9}$$

satisfies

$$y(T) \le y(0) - 2\pi. \tag{10}$$

As the function $g_0(y + F_2(t))$ is Lipschitz continuous in y, the Cauchy problem

$$\begin{aligned} &cy'(t) + g_0\big(y(t) + F_2(t)\big) = 0, \qquad t \in \mathbb{R} \\ &y(0) = y_0, \end{aligned} \tag{11}$$

has an unique solution $y(t)$.

Let $y(t)$ be a solution of (11). Then as in the previous case the solution $z(t)$ of (11) satisfying $y_0 + 2(n-1)\pi \le z_p \le y_0 + 2n\pi$ is bounded by (7). With respect to (10) and (7) we obtain $y(t + nT) \le y(t) - 2n\pi$ for all $t \in \mathbb{R}$, so that $\lim_{t \to +\infty} y(t) = -\infty$.

Now from (7) it follows that Eq. (4) does not admit T-periodic solutions.

Since (4) does not possess a T-periodic solution in both cases ($i = 1$ or 2), whence

$$c \in \left(0, \frac{T \max_{\xi \in [0,2\pi]} |g_0(\xi)|}{2\pi}\right].$$

From now on, we will denote

$$L_T^p = \Big\{ u : \mathbb{R} \to \mathbb{R}, \text{ measurable}, u(t) = u(t+T) \text{ a.e. in } \mathbb{R},$$

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p \, dt \right)^{1/p} < +\infty \Big\}, \qquad 1 \le p < +\infty,$$

and

$$L_T^\infty = \Big\{ u : \mathbb{R} \to \mathbb{R}, \text{ measurable}, \; u(t) = u(t+T) \text{ a.e. in } \mathbb{R},$$

$$\|u\|_{L^\infty} = \inf_{\substack{\mathscr{N} \\ \text{meas}\,\mathscr{N}=0}} \; \sup_{x \in [0,T] \setminus \mathscr{N}} |u(t)| < +\infty \Big\}.$$

Let $F \in L_T^1$ and let us consider the equation

$$cy'(t) + g_0(y(t) + F(t)) = 0. \tag{12}$$

Then

$$\mathscr{P} = \{ F \in L_T^1 : \text{Eq. (12) has a } T\text{-periodic solution} \} \tag{13}$$

is a proper subset of L_T^1.

Let us show that $\mathscr{P}$ it is also closed in L_T^1. Let $\{F_n\}_0^\infty \subset L_T^1$ be a sequence converging in L_T^1 to F and let y_n be a sequence of T-periodic solutions of the equation

$$cy_n'(t) + g_0(y_n(t) + F_n(t)) = 0. \tag{14}$$

As the function g_0 is 2π-periodic, it is possible to assume

$$|y_n(0)| \le 2\pi \tag{15}$$

without loss of generality.

From (14) one has

$$\|y_n'\|_{L^\infty} \le \frac{\max_{\xi \in [0, 2\pi]} |g_0(\xi)|}{c}, \tag{16}$$

and it implies that the functions y_n are equicontinuous for each c, and y_n are also equibounded because of (15) and (16). Using the Arzelà–Ascoli theorem and passing to the limit we get that $F \in \mathscr{P}$, which gives that $\mathscr{P}$ is closed.

As $\mathscr{P}$ is a closed, proper subset of L_T^1, and C_T^1 is a dense subset of L_T^1, there exists $F_0 \in C_T^1 \cap (L_T^1 \setminus \mathscr{P})$, which means that taking $F = F_0$ in (12), it does not have a T-periodic solution. Using the fact that $F_0 \in C_T^1$, substituting $x(t) = y(t) + F(t)$, one can rewrite (12) as

$$cx'(t) + g_0(x(t)) = cF_0'(t).$$

If we denote cF_0' by $\tilde{f}_0$ in the previous equation (point out that $\int_0^T F_0' = 0$), we obtain that (2) does not have T-periodic solutions for $\tilde{f} = 0$. Obviously, it is possible to repeat the previous procedure with $F_0 \in C_T^\infty$ to show even that there exists a function $\tilde{f}_0 \in C_T^\infty$ such that (2) does not have T-periodic solutions if $\bar{f} = 0$.

Suppose, by contradiction, that for all $\delta > 0$ there exists $|\bar{f}| < \delta$ such that (2) has a T-periodic solution. Then there exist sequences $\bar{f}_n \to 0$ and $x_n \subset C_T^1$ satisfying

$$cx_n'(t) + g_0(x_n(t)) = \tilde{f}_0(t) + \bar{f}_n. \tag{17}$$

Let us multiply (17) by x_n' and integrate from 0 to T. Using that $x_n(0) = x_n(T)$ and the Cauchy–Schwartz inequality one gets

$$\|x_n'\|_{L_T^2} \leq \frac{1}{c}\|\tilde{f}\|_{L_T^2}.$$

Thus the x_n are equicontinuous and due to 2π-periodicity of g_0 we can assume that $|x_{\mathrm{n}}(0)| \leq \pi$, so we obtain that x_n are also equibounded (in $\|\cdot\|_{C_T}$). Using the Arzelà–Ascoli Theorem it is possible to select a convergent subsequence; let $x = \lim_{n_k \to \infty} x_{n_k}$. Since $x_{n_k}(t) = x_{n_k}(0) + \int_0^t (\tilde{f}_0(\tau) + \bar{f}_{n_k} - g_0(x_{n_k}(\tau)))\, d\tau$ passing to the limit yields that $x(t)$ is a T-periodic solution of (2) with $\bar{f} = 0$, but this is a contradiction.

Now it remains to prove that there exists an unbounded convex set $\mathscr{A} \subset C_T$ with nonempty interior, such that if $\tilde{f} \in \mathscr{A}$ and Eq. (2) has a solution then $\bar{f} \neq 0$. Let us suppose that $\max_{\xi \in [0,2\pi]}|g_0(\xi)| = |\min_{\xi \in [0,2\pi]} g_0(\xi)|$, for it not, $\max_{\xi \in [0,2\pi]}|g_0(\xi)| = |\max_{\xi \in [0,2\pi]} g_0(\xi)|$, use F_2 instead of F_1 in the remainder of this proof. Due to the continuous dependence of the solution of (6) on the right-hand side, we can see that there exists $\delta > 0$ such that for all $F \in L_T^1 : \|F - F_i\|_{L_T^1} < \delta$, $i = 1, 2$, the Cauchy problem (6) does not admit a T-periodic solution. Indeed, consider that for all $\delta > 0$ there exists F satisfying $\|F - F_1\|_{L_T^1} < \delta$ such that $F \in \mathscr{P}$. Then, since $\mathscr{P}$ is closed, F_1 belongs to $\mathscr{P}$, which is a contradiction. Hence

$$\mathscr{B} = \left\{F \in L_T^1 : \|F - F_i\|_{L_T^1} < \delta, i = 1, 2\right\} \cap C_T^1$$

is a subset of $L_T^1 \setminus \mathscr{P}$ nonempty, since C_T^1 is a dense subset of L_T^1. Let us take any $G_1, G_2 \in \mathscr{B}$. Since $\{F \in L_T^1 : \|F - F_1\| < \delta\}$ is convex, $\lambda G_1 + (1 - \lambda)G_2 \in \{F \in L_T^1 : \|F - F_1\|_{L_T^1} < \delta\}$ for all $\lambda \in [0, 1]$; on the other hand $G_1, G_2 \in C_T^1$ and so $\lambda G_1 + (1 - \lambda)G_2 \in C_T^1$ for all $\lambda \in \mathbb{R}$. Hence $\lambda G_1 + (1 - \lambda)G_2 \in \mathscr{B}$ for all $\lambda \in [0, 1]$, and $\mathscr{B}$ is convex.

Now let us take

$$\mathscr{A} = \left\{\tilde{f} \in \tilde{C}_T : \tilde{f} = \left.\frac{dF}{dt}\right|_{[0,T]}, \text{where } F \in \mathscr{B}\right\};$$

since $\mathscr{B}$ is convex, so is $\mathscr{A}$.

Let us prove that $\mathscr{A}$ is unbounded; suppose that it is not true. Then there exists $K > 0$ such that for all $\tilde{f} \in \mathscr{A} : \|\tilde{f}\|_{\tilde{C}_T} \leq K$ and then, for all $F \in \mathscr{B}$ we have $\|F'\|_{\tilde{C}_T} \leq K$. Consider $\{G_n\}_{n=1}^{\infty} \subset C_T^1$, such that $G_n(t) = F_1(t)$ for all $t \in [0, T - \frac{T}{n+1}]$. Then there exists $t_0 \in [T - \frac{T}{n+1}, T]$ such that $G_n'(t_0) \leq (0 - \beta_1(T - T/(n+1)))/(T - (T - T/(n+1))) = -\beta_1(T - \frac{T}{n+1})n < 0$. So for every K we can find n, such that $\|G_n'\|_{C_T} \geq K$. Thus $\mathscr{A}$ is unbounded.

At the end let us prove that $\mathscr{A} \subset C_T$ has a nonempty interior (i.e., contains some ball in $\|\cdot\|_{C_T}$ norm). With respect to the definition of $\mathscr{A}$, the set $\mathscr{A} \subset C_T$ has nonempty interior if $\mathscr{B} \subset C_T^1$ has. Let $G \in \mathscr{B}$ and

$$\|F - G\|_{C_T^1} < \frac{\delta - \|G - F_1\|_{L_T^1}}{T}.$$

Due to triangle inequality, $\|F - F_1\|_{L_T^1} \leq \|F - \|_{L_T^1} + \|G - F_1\|_{L_T^1}$ and since $\|F - G\|_{L_T^1} \leq T\|F - G\|_{C_T^1}$, we obtain $\|F - F_1\|_{L_T^1} < \delta$. Thus

$$\left\{ F \in C_T^1 : \|F - G\|_{C_T^1} < \frac{\delta - \|G - F\|_{C_T^1}}{T} \right\} \subset \mathscr{B}.$$

So that the set $\mathscr{B} \subset C_T^1$ has nonempty interior. ∎

Remark 2.1. It is worth noting that when using the previous lemma it is possible to obtain new qualitative information about the structure of the range of the operator $H_1 : C_T^2 \to C_T$, $x \to mx'' + g(x')$, with g satisfying [P] (see [4, 6] for previous studies). Indeed, by substitution $v = x'$, the BVP

$$\begin{aligned} &mx''(t) + g(x'(t)) = f(t), \qquad && t \in [0, T], \\ &x(0) = x(T), && x'(0) = x'(T) \end{aligned} \tag{18}$$

is reduced to

$$\begin{aligned} &v'(t) + g(v(t)) = f(t), \qquad && t \in [0, T], \\ &v(0) = v(T), && \int_0^T v(t)\, dt = 0. \end{aligned} \tag{19}$$

Due to [4, 6] there exists $s : C_T \to \mathbb{R}$, such that the BVP (19) possesses a solution if and only if $f = \tilde{f} + s(\tilde{f})$. Now let us suppose that g satisfies [P] and $2\pi / \max_{\xi \in [0, 2\pi]} |g(\xi)| \leq T$. Then, from Lemma 2.1, it follows that there exists a convex, unbounded set $\mathscr{A} \subset \tilde{C}_T$, having nonempty interior, such that $\tilde{f} \in \mathscr{A}$ and $v' + g(v) = f$ has a T-periodic solution (i.e., $v(0) = v(T)$) implies $s(\tilde{f}) \neq 0$. Since $s : \tilde{C}_T \to \mathbb{R}$ is continuous (see [4, 6]) and $\mathscr{A}$ is connected, $s(\mathscr{A}) > 0$ or $s(\mathscr{A}) < 0$. This result can be formulated in the following corollary.

COROLLARY 2.1. *Let $2\pi m/\max_{s\in\mathbb{R}} g_1(s) \le T$ and g_1 satisfy* [P]. *Then there exists an unbounded and convex set $\mathscr{A} \subset \tilde{C}_T$ such that if $\tilde{f} \in \tilde{\mathscr{A}}$ and* (18) *possesses a solution then $s(\tilde{f}) \neq 0$.*

Moreover, $s(\mathscr{A}) \subset (0, +\infty)$ or $s(\mathscr{A}) \subset (-\infty, 0)$.

THEOREM 2.1. *Let g_0 be a Lipschitz continuous function satisfying* [P], *$c \in (0, T\max_{\xi\in[0,2\pi]}|g_0(\xi)|/2\pi]$, and γ_1 a bounded continuous function. Let $\tilde{f}_0 \in \tilde{C}_T$ and $\delta > 0$ be from Lemma* 2.1. *Then for any fixed $\bar{f} \in (-\delta, \delta)$ there exists $m_0 > 0$, $\varepsilon_0 > 0$, such that for $0 < |m| < m_0$, $0 \le \varepsilon < \varepsilon_0$, the equation*

$$mx''(t) + cx'(t) + \varepsilon\gamma_1(x'(t)) + g_0(x(t)) = \tilde{f}_0(t) + \bar{f} \qquad (20)$$

does not admit a T-periodic solution.

Proof. We will use an indirect argument (cf. [10]). Let $\{m_n\}$ and $\{\varepsilon_n\}$ be sequences such that $|m_n| > 0$, $\varepsilon_n > 0$, $m_n \to 0$, $\varepsilon_n \to 0$ and consider $x_n \in C_T^2$ solutions of

$$m_n x_n''(t) + cx_n'(t) + \varepsilon_n\gamma_1(x_n'(t)) + g_0(x_n(t)) = \tilde{f}_0(t) + \bar{f}, \quad (21)$$

where $\bar{f} \in (-\delta, \delta)$.

We can use again the Arzelà–Ascoli Theorem to show that there exists a subsequence of x_{n_k} converging uniformly to x in C_T, and to extend T-periodically to x_{n_k} and x in $C(\mathbb{R})$. Let us multiply (21) by a test function $\varphi \in C_0^\infty(\mathbb{R})$ and integrate by parts

$$m_{n_k}\int_{\mathbb{R}} x_{n_k}\varphi'' + \varepsilon_{n_k}\int_{\mathbb{R}}\gamma_1(x_{n_k}')\varphi + \int_{\mathbb{R}}\left(-cx_{n_k}\varphi' + g_0(x_{n_k})\varphi\right)$$

$$= \int_{\mathbb{R}}\left(\tilde{f}_0 + \bar{f}\right)\varphi.$$

As $|\int_{\mathbb{R}}\gamma_1(x_k')\varphi| \le \frac{1}{c}(\sqrt{\text{meas supp}_\varphi}\sup_{\xi\in\mathbb{R}}|\gamma_1| + \sqrt{\int_{\text{supp}_\varphi}\varphi^2})$, where supp_φ is the support of φ, if k tends to infinity, we get

$$\int_{\mathbb{R}}(-cx\varphi' + g_0(x)\varphi) = \int_{\mathbb{R}}\left(\tilde{f}_0 + \bar{f}\right)\varphi.$$

So the function $x(t)$ is a T-periodic solution of Eq. (2) which contradicts Lemma 2.1. ∎

COROLLARY 2.2. *Let $c \in (0, \frac{T}{2\pi}]$, $\tilde{f}_0 \in \tilde{C}_T$ and $\delta > 0$ be from Lemma* 2.1. *Then for any fixed $\bar{f} \in (-\delta, \delta)$ there exists $m_0 > 0$, $\varepsilon_0 > 0$ such that for all $0 < |m| < m_0$, $0 < \varepsilon < \varepsilon_0$ the equation*

$$mx''(t) + cx'(t) + \varepsilon\gamma_1(x'(t)) + \sin(x(t)) = \tilde{f}_0(t) + \bar{f}$$

does not admit a T-periodic solution.

Remark 2.2. Let us observe that this corollary includes [10, Theorem B] (taking $m > 0$ and $\bar{f} \equiv 0$).

COROLLARY 2.3. *Let us consider*

$$x''(t) + \alpha x'(t) + \gamma_1(x'(t)) + g_0(x(t)) = \tilde{f}(t) + \bar{f}, \tag{22}$$

where the g_0 *satisfy* $[P]$ *and* γ_1 *is a bounded continuous function. So for all* $n \in \mathbb{N}$ *there exists* $k > 0$ *such that if* $\alpha > k$, $\max_{\xi \in [0,2\pi]}|g_0(\xi)| > k$, *and*

$$0 < \frac{2\pi\alpha}{\max_{\xi \in [0,2\pi]}|g_0(\xi)|} < T$$

then there exists $\tilde{f}$ *such that for all* $\bar{f}$, $|\bar{f}| < n$, *Eq.* (22) *has no T-periodic solution.*

Proof. Taking $k > 0$ and dividing (22) by k,

$$\frac{1}{k}x''(t) + \frac{\alpha}{k}x'(t) + \frac{1}{k}\gamma_1(x'(t)) + \frac{1}{k}g_0(x(t)) = \frac{\tilde{f}}{k} + \frac{\bar{f}}{k}. \tag{23}$$

If

$$\alpha \in \left(0, \frac{T\max_{\xi \in [0,T]}|g_0(\xi)|}{2\pi}\right]$$

then, due to Theorem 2.1, there exist $m_0, \varepsilon_0, \delta, \tilde{f}$ such that for $0 < \frac{1}{k} < \min\{m_0, \varepsilon_0, \frac{\delta}{n}\}$, Eq. (23) does not have T-periodic solution. Obviously the same holds also for (22). ∎

In the next lemma we will prove an a priori bound, which we use to show that the range of the operator corresponding to the problem (1) does not include the range when $g_0 \equiv 0$ (see [6, Lemma 1; 9, Theorem 2]).

LEMMA 2.2. *Let* $g_1 : \mathbb{R} \to \mathbb{R}$ *be a continuous function*, $g_0 : \mathbb{R} \to \mathbb{R}$ *be a continuous bounded function, and* $\tilde{f} \in C_T$. *If x is a solution of* (1), *then*

$$\|x'\|_{C_T} \le \frac{\sqrt{T}}{m\sqrt{12}}\left(\sqrt{T}\sup_{\xi \in \mathbb{R}}|g_0(\xi)| + \|\tilde{f}\|_{L_T^2}\right). \tag{24}$$

Proof. Multiply the equation from (1) by x'' and integrate from 0 to T. Because of the periodic boundary conditions and the Cauchy–Schwartz inequality, we get

$$\|x''\|_{L_T^2} \le \frac{1}{m}\left\|\tilde{f}(x) - g_0(x)\right\|_{L_T^2}.$$

Due to periodic boundary conditions, we have $x \in C_T^1 \subset W_T^{1,2}$ and $\int_0^T x'(t)\,dt = 0$. So we can use the Sobolev inequality [8, Proposition 1.3] and estimate $\|g_0(x)\|_{L_T^2} \leq \sqrt{T}\sup_{\xi \in \mathbb{R}}|g_0(\xi)|$, to get the desired bound (24). ∎

THEOREM 2.2. *Let g_0 verify* $[P]$.

Then there exists a real number m and a nonlinear, continuous function g_1, such that the range $\mathscr{R}_1$ of the operator

$$H_1 : C_T^2 \to C_T, \qquad x \mapsto mx'' + g_1(x')$$

is not a subset of the range $\mathscr{R}$ of the operator

$$H : C_T^2 \to C_T, \qquad x \mapsto mx'' + g_1(x') + g_0(x).$$

Proof. Let $\gamma_1 \in C(\mathbb{R}, \mathbb{R})$ be nonconstant and bounded, $c \in (0, T\max_{\xi \in [0,2\pi]}|g_0(\xi)|/2\pi]$, and $\tilde{f}_0 \in \tilde{C}_T$, $\delta > 0$ be from Lemma 2.1. Then, due to Theorem 2.1 (note that (20) is a particular case of the equation in (1)), for any $\bar{f} \in (-\delta, \delta)$ there exist $m_0 > 0$, $\varepsilon_0 > 0$, such that for $0 < |m| < m_0$, $0 < \varepsilon < \varepsilon_0$ the equation

$$mx''(t) + cx'(t) + \varepsilon\gamma_1(x'(t)) + g_0(x(t)) = \tilde{f}_0(t) + \bar{f}$$

does not admit a T-periodic solution.

Let us define $\mu : \bar{f} \mapsto m_0$ and $\epsilon : \bar{f} \mapsto \varepsilon_0$, such that if $m > m_0$ or $\varepsilon > \varepsilon_0$, then Eq. (2) has a T-periodic solution.

If $\delta_1 < \delta$, let us show that $\inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f}) > 0$ and $\inf_{\bar{f} \in [-\delta_1, \delta_1]} \epsilon(\bar{f}) > 0$. By definition both must be greater than or equal to zero; conversely, suppose that $\inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f}) = 0$ or $\inf_{\bar{f} \in [-\delta_1, \delta_1]} \epsilon(\bar{f}) = 0$. If $\inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f}) = 0$ then for all $\alpha > 0$ there exists $\bar{f}$ such that $\mu(\bar{f}) < \alpha$. Let $\alpha_n \searrow 0$, choose $\bar{f}_n$, such that $\mu(\bar{f}_n) < \alpha$, and set $m_n = 2\alpha_n > \alpha_n$. Then

$$m_n x_n''(t) + cx_n(t) + g_0(x_n(t)) = \tilde{f}_0 + \bar{f}_n$$

has a T-periodic solution. Multiplying by a test function $\varphi \in C_0^\infty$ and integrating by parts:

$$m_n \int_{\mathbb{R}} x_n \varphi'' - c\int_{\mathbb{R}} x_n \varphi' + \int_{\mathbb{R}} g_0(x_n)\varphi = \int_{\mathbb{R}} \left(\tilde{f}_0 + \bar{f}_n\right)\varphi. \tag{25}$$

Since x_n is equibounded and equicontinuous in $\|\cdot\|_{C_T}$ and $\bar{f}_n \in [-\delta_1, \delta_1]$, due to the Arzelà–Ascoli Theorem and compactness of $[-\delta_1, \delta_1]$ one can pass to the limit obtaining that

$$\int_{\mathbb{R}} (-cx\varphi' + g_0(x)\varphi) = \int_{\mathbb{R}} \left(\tilde{f}_0 + \bar{f}\right)\varphi,$$

where $\bar{f} \in [-\delta_1, \delta_1] \subset (-\delta, \delta)$ and the function $x(t)$ is a T-periodic solution of Eq. (2), which contradicts Lemma 2.1.

Now suppose that

$$\inf_{\bar{f} \in [-\delta_1, \delta_1]} \epsilon(\bar{f}) = 0,$$

and fix $0 < m < \inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f})$. For a sequence of positive numbers $\alpha_n \searrow 0$, choose $\bar{f}_n$, such that $\epsilon(\bar{f}_n) < \alpha_n$ and set $\varepsilon_n = 2\alpha_n > \alpha_n$. Then

$$mx_n''(t) + cx_n'(t) + \varepsilon_n \gamma(x_n'(t)) + g_0(x_n(t)) = \tilde{f}_0 + \bar{f}_n$$

has a T-periodic solution. Multiplying by x'', integrating from 0 to T, and using periodic boundary conditions and the Cauchy–Schwartz inequality one arrives at

$$\|x''\|_{L_T^2} \leq \frac{1}{m} \|\tilde{f}_0\|,$$

so that x_n', x_n are equicontinuous and the x_n' are equibounded in $\|\cdot\|_{C_T}$. Since g_0 is 2π-periodic, we can assume $|x_n(0)| \leq \pi$. Thus the x_n are equibounded. Using the Arzelà–Ascoli Theorem it is possible to find a subsequence convergent in C_T^1; let $x = \lim_{n_k \to \infty} x_{n_k}$. Since

$$mx_{n_k}'(t) = x_{n_k}'(0) + \int_0^t \left(\tilde{f}_0 + \bar{f}_{n_k} - cx_{n_k}'(t) - \varepsilon_n \gamma\left(x_{n_k}'(t)\right) - g_0\left(x_{n_k}(t)\right)\right)$$

and passing to the limit we get that x is a T-periodic solution of

$$mx''(t) + cx'(t) + g_0(x(t)) = \tilde{f}_0 + \bar{f},$$

contrary to Theorem 2.1, because $0 < m < \inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f})$ and $\bar{f} \in [-\delta_1, \delta_1] \supset (-\delta, \delta)$.

Let us fix

$$m < \inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f}) \quad \text{and}$$

$$\varepsilon < \min\left\{ \delta_1 \Big/ \sup_{\xi \in \mathbb{R}} |\gamma_1(\xi)|, \inf_{\bar{f} \in [-\delta_1, \delta_1]} \epsilon(\bar{f}) \right\}.$$

Consequently, define

$$d = \frac{\sqrt{T}}{m\sqrt{12}} \left(\sqrt{T} \max_{\xi \in [0, 2\pi]} |g_0(\xi)| + \|\tilde{f}_0\|_{L_T^2} \right)$$

and $g_1 \in C(\mathbb{R},\mathbb{R})$ such that $g_1(\xi) = c\xi + \varepsilon\gamma_1(\xi)$ for $\xi \in [-d, d]$. Since Lemma 2.2 yields an a priori bound for solutions of (1), $\|x'\|_{L^\infty} \le d$, the solutions of (1) coincide with T-periodic solutions of (20). But owing to Theorem 2.1,

$$\text{(20) does not admit a solution for any } \bar{f} \in [-\delta_1, \delta_1]. \tag{26}$$

Now let us consider the differential equation in (1), with $g_0 \equiv 0$, split $f = \tilde{f} + \bar{f}$. By [4, 6] it is known that for all $\tilde{f} \in \tilde{C}_T$ there exists $\bar{f}$ such that (1), with $g_0 \equiv 0$, has a solution. Now we are going to prove a suitable estimate on $\bar{f}$. Let us integrate the equation in (1) from 0 to T. Then, using the periodic boundary conditions, we get

$$\bar{f} = \frac{1}{T}\int_0^T \{g_1(x'(t)) - cx'(t)\}\,dt.$$

Taking into account that $\|x'\|_{L_T^\infty} \le d$ and $\varepsilon < \delta_1/\max_{\xi \in [-\delta_1, \delta_1]}\gamma_1(\xi)$ we obtain

$$|\bar{f}| \le \max_{\xi \in [-\delta_1, \delta_1]} \varepsilon\gamma_1(\xi) < \delta_1. \tag{27}$$

With respect to (27) we have

$$\tilde{f}_0 + \bar{f} \in \mathscr{R}_1 \Rightarrow \bar{f} \in [-\delta_1, \delta_1],$$

and (26) means

$$\bar{f} \in [-\delta_1, \delta_1] \Rightarrow \tilde{f}_0 + \bar{f} \notin \mathscr{R},$$

which concludes the proof. ∎

Remark 2.3. If $g_1(x') = cx'$ then the range $\mathscr{R}_1 = \{f \in C_T : \bar{f} \equiv 0\}$ (this case has been studied in [10]); $\mathscr{R}_1$ is not possible to express explicitly for nonlinear g_1. This is the reason for which the function g_1 is considered to be near to the linear one in some interval and m is small (note that it can be bounded). It would be interesting to improve the result for general nonlinear g_1 and m arbitrarily large (in [1] the case with general m has been considered).

Remark 2.4. The previous theorem can be written in a more general form.

Let g_0 verify [P], $\gamma_1 \in C(\mathbb{R},\mathbb{R})$ a nonconstant and bounded function, and $c \in (0, T\max_{\xi \in [0,2\pi]}|g_0(\xi)|/2\pi]$.

Then there exist $M, E > 0$ such that for all $0 < |m| < M$, $0 < |\varepsilon| < E$, if $g_1(\tau) = c\tau + \varepsilon\gamma_1(\tau)$ then the range $\mathscr{R}_1$ of the operator

$$H_1 : C_T^2 \to C_T, \qquad x \mapsto mx'' + g_1(x')$$

is not a subset of the range $\mathscr{R}$ of the operator

$$H : C_T^2 \to C_T, \qquad x \mapsto mx'' + g_1(x') + g_0(x).$$

(Take $E = \min\{\delta_1/\sup_{\xi \in \mathbb{R}}|\gamma_1(\xi)|, \inf_{\bar{f} \in [-\delta, \delta_1]}\epsilon(\bar{f})\}$ and $M = \inf_{\bar{f} \in [-\delta_1, \delta_1]} \mu(\bar{f})$.)

2.2. *Local Inclusion*

In this section we show that if one imposes some additional assumptions then the range $\mathscr{R}$ contains an open connected subset of the range $\mathscr{R}_1$.

LEMMA 2.3. *Under the assumptions of the next theorem* (*Theorem* 2.3), *there exists* $\bar{f}_1$ *satisfying*

$$g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi) - \alpha < \bar{f}_1 < g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi)$$

such that $\mathsf{F}_{\bar{f}_1} = 0$ *has a constant solution* $x(t) \equiv \eta_1$ *and the operator* $\mathsf{F}_{\bar{f}_1}$ *has an invertible Fréchet derivative at* $x(t) = \eta_1$; *and there exists* $\bar{f}_2$ *satisfying*

$$g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) < \bar{f}_2 < g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha$$

such that $\mathsf{F}_{\bar{f}_2} = 0$ *has a constant solution* $x(t) \equiv \eta_2$ *and the operator* $\mathsf{F}_{\bar{f}_2}$ *has an invertible Fréchet derivative at* $x(t) \equiv \eta_2$.

Proof. It is clear that if $\tilde{f} \equiv 0$ then, if there exists $\eta \in \mathbb{R}$ such that

$$g_1(0) + g_0(\eta) = \bar{f},$$

then $x(t) \equiv \eta$ is a periodic solution of

$$mx''(t) + g_1(x'(t)) + g_0(x(t)) = \tilde{f} + \bar{f}.$$

A necessary and sufficient condition of the existence of such η (because of the continuity of g_0) is

$$g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) < \bar{f} < g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi).$$

Let us define the operator $\mathsf{F}_{\bar{f}} : C_T^2 \to C_T$, by $x \mapsto mx'' + g_1(x') + g_0(x) - \bar{f}$. The Fréchet derivative of $\mathsf{F}_{\bar{f}}$ with respect to x exists and is given by

$$\mathsf{F}'_{\bar{f}}(x)v = mv'' + g'_1(x')v' + g'_0(x)v \qquad \text{for any } v \in C_T^2.$$

Using this formula it is possible to show that $\mathsf{F}'_{\tilde{f}}: C_T^2 \to \mathscr{L}(C_T^2, C_T)$ is continuous (by $\mathscr{L}(C_T^2, C_T)$ we mean the space of continuous linear mappings $C_T^2 \to C_T$).

From the Fredholm Alternative Theorem (see [3, Theorem 0.1]) it follows that $\mathsf{F}'_{\tilde{f}}(x)$ is an invertible operator if and only if $\mathsf{F}'_{\tilde{f}}(x)v = 0$ has only the trivial solution $v \equiv 0$.

Let us suppose that $g'_1(0) = 0$. Then the equation $\mathsf{F}'_{\tilde{f}}(x)v = 0$ has no trivial solution if and only if $g'_0(\eta_1) = 4\pi^2 n^2 m/T^2$, where $n \in \mathbb{N} \cup \{0\}$. By [P] and $g_0 \not\equiv 0$ we have that g_0 is not constant. Hence

$$\forall \alpha > 0, \exists \hat{\eta}_1, \qquad g_0(\hat{\eta}_1) < \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha \text{ such that } g'_0(\hat{\eta}_1) \neq 0. \tag{28}$$

As the function g_0 is continuously differentiable and 2π-periodic, let ξ_{Min} be such that $g_0(\xi_{\mathrm{Min}}) = \min_{\xi \in [0, 2\pi]} g_0(\xi)$; then $g'_0(\xi_{\mathrm{Min}}) = 0$. If $g'_0(\hat{\eta}_1) = 4\pi^2 n^2 m/T^2$ for some $n \in \mathbb{N} \cup \{0\}$ then, because of the continuity of g'_0, there exists η_1 such that $g'_0(\eta_1) \neq 4\pi^2 n^2 m/T^2$. Set $x(t) \equiv \eta_1$; then $\mathsf{F}'_{\tilde{f}}(x)$ is an invertible mapping.

Analogously,

$$\forall \alpha > 0, \exists \hat{\eta}_2, \qquad g_0(\hat{\eta}_2) < \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha \text{ such that } g'_0(\hat{\eta}_2) \neq 0, \tag{29}$$

and it can be deduced that there exists η_2 such that if $x \equiv \eta_2$ then $\mathsf{F}'_{\tilde{f}}(\hat{\eta}_2)$ is also an invertible mapping.

If $g'_1(0) \neq 0$, the equation

$$mv'' + g'_1(0)v' + g'_0(\eta)v = 0$$

has only the trivial T-periodic solution if and only if $g'_0(\eta) \neq 0$. The last inequality holds true for η_1, η_2 given by (28) and (29), so $\mathsf{F}'_{\tilde{f}}(\eta_i)$, $i = 1, 2$, is an invertible mapping. ∎

THEOREM 2.3. *Let g_0, g_1 be continuously differentiable functions and $g_0 \not\equiv 0$ verifying $[P]$. Then for all α such that*

$$0 < \alpha < \min\left\{ \max_{\xi \in [0, 2\pi]} g_0(\xi), - \min_{\xi \in [0, 2\pi]} g_0(\xi) \right\},$$

there exists $\varepsilon_\alpha > 0$ such that for all $\|\tilde{f}\|_C < \varepsilon_\alpha$ the equation

$$mx''(t) + g_1(x'(t)) + g_0(x(t)) = \tilde{f} + \bar{f} \tag{30}$$

has a T-periodic solution if

$$g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha < \bar{f} < g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi) - \alpha. \quad (31)$$

Proof. In the proof we use the fact that for the functions $f \in C_T$ having $\bar{f} \equiv 0$ we can construct the solution explicitly and then we use the Local Inversion Mapping Theorem [3, Theorem 1.2].

Define the operator $\mathsf{F}_{\bar{f}_1} : C_T^2 \to C_T$, defined by the map $x \mapsto mx'' + g_1(x') + g_0(x) - \bar{f}_1$. Because of Lemma 2.3, there exists

$$g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi) - \alpha < \bar{f}_1 < g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi)$$

such that

$$\mathsf{F}_{\bar{f}_1} = 0$$

has a constant solution $x(t) \equiv \eta_1$, and the operator $\mathsf{F}_{\bar{f}_1}$ has an invertible Fréchet derivative at $x(t) = \eta_1$. The inverse of the Fréchet derivative of $\mathsf{F}_{\bar{f}_1}$ is continuous due to the Banach Open Theorem (see [11]). Using the Local Inversion Mapping Theorem, there exists $\varepsilon_1(\alpha) > 0$ such that for all $\tilde{f}$, $\|\tilde{f}\|_C < \varepsilon_1$ there exists a solution of

$$mx''(t) + g_1(x'(t)) + g_0(x(t)) = \tilde{f} + \bar{f}_1.$$

Analogously there exist

$$g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) < \bar{f}_2 < g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha$$

and $\varepsilon_2(\alpha) > 0$ such that for all $\tilde{f}$, $\|\tilde{f}\|_C < \varepsilon_2$, the equation

$$mx''(t) + g_1(x'(t)) + g_0(x(t)) = \tilde{f} + \bar{f}_2$$

has a solution.

Let us take $\varepsilon_\alpha = \min\{\varepsilon_1(\alpha), \varepsilon_2(\alpha)\}$. Then due to [4, Proposition 2.1] Eq. (30) has a solution for all f satisfying $\|\tilde{f}\| < \varepsilon_\alpha$ and

$$g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha < \bar{f} < g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi) - \alpha.$$

∎

THEOREM 2.4. *There exists $\varepsilon > 0$ such that if $f \in \mathscr{R}_1$ and $\|\tilde{f}\|_C < \varepsilon$, then also $f \in \mathscr{R}$.*

Proof. The equation

$$mx''(t) + g_1(x'(t)) = s(0)$$

possesses an unique (see [4, Theorem 3.4]) periodic solution, which is constant. So we get

$$s(o) = g_1(0).$$

From Theorem 2.3 we know that for

$$0 < \alpha < \min\left\{ \max_{\xi \in [0, 2\pi]} g_0(\xi) - \min_{\xi \in [0, 2\pi]} g_0(\xi) \right\},$$

Eq. (30) has a periodic solution for all $\bar{f}$ satisfying condition (31) and $\|\tilde{f}\|_C \leq \varepsilon_\alpha$. Due to the continuity of s (see [4, Theorem 3.4; 6, Theorem 1]) there exists $\delta > 0$ such that

$$g_1(0) + \min_{\xi \in [0, 2\pi]} g_0(\xi) + \alpha < s(\tilde{f}) < g_1(0) + \max_{\xi \in [0, 2\pi]} g_0(\xi) - \alpha \quad (32)$$

for all $\|\tilde{f}\|_C < \delta$. Hence, for all $\|\tilde{f}\|_C < \min\{\varepsilon_\alpha, \delta\}$ we get that $\tilde{f} + s(\tilde{f}) \in \mathscr{R}_1$ belongs also to $\mathscr{R}$. ∎

COROLLARY 2.4. *For all* $m \neq 0$, $c \in \mathbb{R}$, *and* $\alpha \in (0, 1)$, *there exists* $\varepsilon > 0$ *such that for all* $\tilde{f}$, $\|\tilde{f}\|_C < \varepsilon$ *the equation*

$$mx'' + cx' + \sin(x) = \tilde{f} + \bar{f}$$

has a T-periodic solution if

$$|\bar{f}| < 1 - \alpha.$$

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